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Propagation and bouncing period of VLF waves through inhomogeneous magnetosphere

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The effects of small scale field aligned and non-field aligned plasma inhomogeneities on the VLF wave propagation through the magnetosphere have been studied. It has been shown that the plasma inhomogeneities help the VLF wave guiding in magnetosphere. The expressions for phase and group refractive indices and bounce period have been obtained It has been demonstrated that the VLF wave bounce period decreases due to the non-field aligned plasma inhomogeneities

1. INTRODUCTION

VLF wave propagate through the magnetosphere in two different modes : ducted and non ducted While the ducted mode of propagation requires an irregular structure of the ambient plasma medium, that is field aligned irregularities of enhanced or reduced ionization (Smith *et al* 1960; Walker 1971, Okuzawa & Yamanaka 1973), the non-ducted mode is possible in a smooth medium in which there are only large scale electron density variations A theoretical and experimental study of non-ducted VLF waves after propagation through the magnetosphere has been made by Cerisier (1973). VLF observations made by OGO-I satellite (Smith & Angerami 1968) yields that non-ducted propagation occurs at low latitudes where as ducted propagation is observed mainly at $L > 2.5$

The experimental observations of plasma irregularities (Helliwell 1965, Smith & Angerami 1968) motivated us to study the effect of small scale irregularities on a whistler wave which is being guided along a magnetic field lines of force in the magnetosphere The treatment of the present paper follows the approach of Budden (1959) In the present paper the dipole model of the earth's magnetic field and the constant electron density along magnetic field line have been considered (Thorne & Kennel 1967) In section 2 we have studied the VLF waves propagation through inhomogeneous magnetosphere by taking suitable expressions for the phase refractive index which accounts for the field aligned and non-field aligned plasma irregularities The expressions for group refractive index and bouncing time of the VLF wave have been derived in sections 2 and 3 respectively In section 4 we have discussed the numerical results and the conclusion has been given in section 5.

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2. Dispersion Relation and Group Velocity

For simplicity we shall confine our attention to the whistler mode wave propagation in cold and collisionless magnetospheric plasma, satisfying the condition that the ratio of plasma density to magnetic field intensity is constant at the equator (Thorne & Kennel 1967) This constant density model is a valid assumption in the magnetospheric region (Thorne & Kennel 1967) Although the validity of the model down to the ionospheric height may be questioned but we have used this model as it greatly simplifies the analysis Using this model, Singh & Singh (1970) studied the propagation characteristics of VLF waves in the magnetosphere Their study did not include the effect of small scale plasma inhomogeneities Here in the present paper we have included the effect of plasma inhomogeneities on VLF wave propagation. In order to include this feature we have chosen the refractive index for the whistler mode given by Budden (1959)

$$\mu^2 = \frac{\omega_p{}^2/\omega_c\omega}{1-\frac{\beta M}{2}\frac{\omega_p{}^4}{\omega^4}\cdot\frac{\omega}{\omega_c}+\frac{\beta l}{2}\frac{\omega_p{}^4}{\omega^4}\frac{\omega^3}{\omega_c{}^3}+\frac{\beta Q}{2}\frac{\omega_p{}^2}{\omega^2}\cdot\frac{\omega}{\omega_c}} \qquad (1)$$

where

ω_p is the plasma frequency,

ω_c is the electrons cyclotron frequency,

ω is the wave frequency,

$\beta = \left(\frac{<\Delta N>}{N}\right)^2$, the angular bracket is used for the average,

N is the average electron number density,

ΔN is the fluctuation in the electron number density due to irregularities,

M is a measure of the elongation of the irregularities in the transverse direction (Budden 1959, equation B 17) and is a dimensionless parameter,

l is the measure of the elongation of the irregularities in the magnetic field direction (Budden 1959, equation B 16) and is a dimensionless parameter

$Q = M+l$

Assuming the dipole nature of the geomagnetic field, the variation of the gyrofrequency along the field line in terms of the equatorial gyrofrequency is expressed as

$$\omega_c = \frac{(\omega_c)_e}{f(\phi)} \qquad \ldots \quad (2)$$

where the suffix e refers to the equatorial value, ϕ is the geomagnetic latitude and $f(\phi)$ is given by

$$f(\phi) = \frac{\cos^6\phi}{(1+3\sin^2\phi)^{\frac{1}{2}}} \quad \ldots \quad (3)$$

Using the above equations, we obtain the phase refractive index expression for the whistler mode propagation valid in the ionospheric and magnetospheric region as follows :

$$\mu = \left[\frac{(\omega_p^2/\omega_c\omega)_e f(\phi)}{1-\frac{\beta M}{2}\left(\frac{\omega_p^4}{\omega_c}\right)_e\cdot\frac{f(\phi)}{\omega^3}+\frac{\beta l}{2}\left(\frac{\omega_p^4}{\omega_c^3}\right)_e\cdot\frac{f^3(\phi)}{\omega}+\frac{\beta Q}{2}\left(\frac{\omega_p^2}{\omega_c}\right)_e\cdot\frac{f(\phi)}{\omega}}\right]^{\frac{1}{2}} \quad (4)$$

The group refractive index, for the wave packet, is defined in terms of the phase refractive index μ. The contribution of the last two terms of the denominator is very small for whistler mode propagation and therefore, neglecting these terms; we obtain an expression for the group refractive index as :

$$\mu' = \frac{1}{2}\left\{\left(\frac{\omega_p^2}{\omega_c}\right)_e\frac{f(\phi)}{\omega}\right\}^{\frac{1}{2}}\cdot\frac{1-4\eta}{(1-\eta)^{3/2}} \quad \ldots \quad (5)$$

where

$$\eta = \frac{\beta M}{2}\cdot\left(\frac{\omega_p^4}{\omega}\right)_e\frac{f(\phi)}{\omega^3}$$

Using eqn (5) and magnetospheric plasma parameters the group refractive index has been computed and results have been discussed in section 4.

The wave energy in the anisotropic medium, propagates along the ray direction with the group velocity $v_g = \frac{c}{\mu'\cos\alpha}$ at an angle α with geomagnetic field direction. For the magnetospheric propagation of VLF waves in the whistler mode we have taken, the angle α is zero. Using eqn. (5) we obtain an expression of the group velocity as

$$v_g = c\left[\left(\frac{\omega_c}{\omega_p}\right)_e^2\cdot\left(\frac{\omega}{\omega_c}\right)_e\frac{4(1-\eta)^3}{f(\phi)(1-4\eta)^2}\right]^{\frac{1}{2}} \quad \ldots \quad (6)$$

where c is the velocity of light. From this expression we easily find that the group velocity is zero when $\eta = 1$. At this stage the propagation is not possible. For the wave propagation the following inequality should be satisfied (i.e., $\eta < 1$),

$$\omega > \left[\frac{\beta M}{2}\left(\frac{\omega_p^4}{\omega_c}\right)_e f(\phi)\right]^{1/3}.$$

If $\eta > 1$, the group velocity becomes imaginary and again there is no propagation. Also, we see that the group velocity becomes infinity at $\eta = \frac{1}{4}$. Since the maximum value of the group velocity can be equal to c (velocity of light in free space) in this situation the approximation in eq. (4) fails. This shows that there will be a limit posed on η and consequently on the irregularity parameters for the given wave frequency and medium parameters

3 Bounce Period of VLF Waves

The bounce period of the VLF wave is defined as

$$T = 4 \int_0^{\phi_r} \frac{\mu' ds}{c} \qquad \ldots \quad (7)$$

where ds is the path element along the geomagnetic field line (ray path) and ϕ_r is the geomagnetic latitude from which the VLF whistlers are reflected back In order to obtain the propagation time for the whistlers on the ground ϕ_r is replaced by ϕ_g, ϕ_g being the geomagnetic latitude where the particular geomagnetic field line meets the ground. The VLF waves, starting with finite wave normal angle from the equatorial region, may not reach the ground because of the magnetic field gradient and may bounce back and forth in the vicinity of equator For numerical evaluation, we have assumed the geomagnetic field variation of dipole nature. In such a case the path length ds in the magnetic field direction is given by

$$ds = \frac{R_E}{\cos^2\phi} . (1+3\sin^2\phi)^{\frac{1}{2}} . \cos\phi \, d\phi \qquad . \quad (8)$$

where R_E is radius of the earth. Eqs (7) and (8) directly lead to

$$T = \frac{2R_E}{c\cos^2\phi_g} . \left(\frac{\omega_p^2}{\omega_c \omega}\right)_e^{\frac{1}{2}} \int_0^{\phi_g} \frac{(1-4\eta).\cos^4\phi(1+3\sin^2\phi)^{1/4}}{(1-\eta)^{3/2}} d\phi \qquad \ldots \quad (9)$$

Since η is a function of ϕ, it would be difficult to integrate the eq. (9) exactly. Assuming η as a small parameter, we can use the binomial expansion in he denominator of eq. (9) and the expression for T may be rewritten as

$$T = \frac{2R_E}{c\cos^2\phi_g} \left(\frac{\omega_p^2}{\omega_c \omega}\right)_e^{\frac{1}{2}} \int_0^{\phi_g} \left(1-\frac{5}{2}\eta\right) \cos^4\phi \, (1+3\sin^2\phi)^{1/4} d\phi \qquad \ldots \quad (10)$$

To obtain the analytical results we shall use the following relations

$$(1+3\sin^2\phi) = 5/2(1-0.6\cos 2\phi) \qquad \ldots \quad (11)$$

$$(1+3\sin^2\phi)^{1/4} = 1.26(1-0.15\cos 2\phi) \qquad .. \quad (12)$$

$$(1+3\sin^2\phi)^{-1/4} = (1\ 26)^{-1}(1+0.15\cos 2\phi) \qquad \ldots \quad (13)$$

In deriving the last two relations, we have used the binomial expansion upto first order In actual computation at the equator we have taken $\phi = 0$ and in the immediate vicinity of the equator $\sin\phi \simeq \phi$ has been used. However, in the non equatorial region we have made use of the binomial expansions given in eqs (12) and (13). With these simplifications, eq (10) is written as :

$$T = \frac{2R_E}{c\cos^2\phi_g}\cdot\left(\frac{\omega_p^2}{\omega_c\omega}\right)^{\frac{1}{2}}_e (1.26)\left[\int_0^{\phi_g}\cos^4\phi(1-0.15\cos 2\phi)d\phi\right.$$

$$\left.-(1.26)^{-2}.\ \frac{5}{4}\frac{\beta M}{\omega^3}\cdot\left(\frac{\omega_p^4}{\omega_c}\right)_e\int_0^{\phi_g}\cos^{10}\phi\,(1+0.15\cos 2\phi)d\phi\right] \quad \ldots \quad (14)$$

Integrating this equation, we get,

$$T = \frac{2R_E}{c\cos^2\phi_g}\left(\frac{\omega_p^2}{\omega_c\omega}\right)^{\frac{1}{2}}_e (1.26)\left[\left\{X-0.15X_1+0.15\right.\right.$$

$$\left.\left\{-\frac{1}{6}\sin\phi_g\cos^5\phi_g+\frac{1}{6}X\right\}\right\}-(1.26)^{-2}.\ \frac{5}{4}\frac{\beta M}{\omega^3}\left(\frac{\omega_p^4}{\omega_c}\right)_e$$

$$\left\{Y+\frac{63}{80}X_2+0.15\left\{\frac{1}{12}\cos^{11}\phi_g\sin\phi_g+\frac{11}{12}\left(Y+\frac{63}{80}X_1\right)\right\}\right.$$

$$\left.\left.-0.15\left\{-\frac{1}{12}\cos^{11}\phi_g\sin\phi_g+\frac{1}{12}\left(Y+\frac{63}{80}X_1\right)\right\}\right\}\right] \quad \ldots \quad (15)$$

where

$$X = \frac{1}{4}\cos^3\phi_g\sin\phi_g+\frac{3}{8}\cos\phi_g\sin\phi_g+\frac{3}{8}\phi_g$$

$$X_1 = \frac{1}{6}\cos^5\phi_g\sin\phi_g+\frac{5}{6}X$$

$$X_2 = -\frac{1}{6}\cos^5\phi_g\sin\phi_g+\frac{5}{6}X$$

$$Y = \frac{1}{10}\cos^9\phi_g\sin\phi_g+\frac{9}{80}\cos^7\phi_g.\sin\phi_g.$$

In the first bracket of eq. (15), the terms are independent of electron density irregularities, while the second bracket shows a reduction in the bounce period due to the presence of plasma inhomogeneities.

4. Numerical Results and Discussions

All the calculations presented in this section have been performed for $L = 4$ in order to compare them with available experimental results. The variations of phase refractive index and group refractive index have been calculated for various frequencies along the geomagnetic lines of force. The plasma and gyrofrequencies corresponding to $L = 4$ have been chosen from the model given by Matsumoto & Kimura (1971). In figures (1) and (2) we have plotted the variation of phase refractive index (eq. (1)) with the geomagnetic latitude ϕ, corresponding to 1, 3, 5 and 10 kHz and for various values of plasma inhomogeniety parameters. From figures (1) and (2) it is seen that for small values of β there is an appreciable change in the dielectric constant. The smallest values of β used for frequencies 1, 3, 5 and 10 kHz are 10^{-10}, 10^{-8}, 10^{-7} and 10^{-6} respectively. Values of β lower than these for corresponding frequencies have negligible effect on the phase refractive index. This is due to the fact that the second and other terms containing β in the denominator of eq. (4) becomes much smaller than 1. Walker (1966a), while studying the ducted whistler mode propagation has shown that the minimum value of $\Delta N/N$ for whistler guiding are of the order of -5×10^{-3}, -2×10^{-3}, 1×10^{-3} and 2×10^{-3} for frequencies of 1, 5, 7 and 10 kHz respectively. The experimental measurements of Nishizaki & Matuura (1971) claims that the maximum value of β should not increase above 0.01 which is realistic for magnetospheric studies. In the case of non field aligned plasma inhomogeneities, the phase refractive index increases with increase of the irregularity parameters and is minimum when $\beta = 0$. From figures (1) and (2) we also find that, when the wave travels from higher latitudes to lower latitudes the phase refractive index changes appreciably for various values of β at different frequencies. At lower latitude the phase refractive index starts tending to infinity and propagation of electromagnetic wave is not possible. The phase refractive index plot for 10 kHz frequency for $\beta = 10^{-5}$ is instructive. As the wave travels down the geomagnetic lines of force the phase refractive index decreases and may approach the reflection point.

In figures (3) and (4) we have shown the variation of phase refractive index (eq. 4) with the geomagnetic latitude in the case of completely field aligned plasma blob ($M = 0$, $Q = 1$). In this case the phase refractive index decreases with the increase of β. The corresponding phase velocity of the wave is found to increase with the increase of β and shows a peak around certain latitude. For example, in the case of wave frequency of 3 kHz at $\beta = 10^{-1}$ the maximum of phase refractive index appears at 30° latitude and with increasing values of β the peak is found to shift towards higher latitudes. Thus we see that the field aligned inhomogeneities help in guiding the VLF wave along the field lines. We have shown the variation of group refractive index with the geomagnetic latitude (figures 5 and 6) in the absence as well as in the presence of non-aligned plasma inhomogeneities ($M \neq 0$, in eq. 5) The group refractive index decreases with the increase of the inhomogeneity strength β. When β increases and the frequency decreases then the group refractive index starts showing a peak For

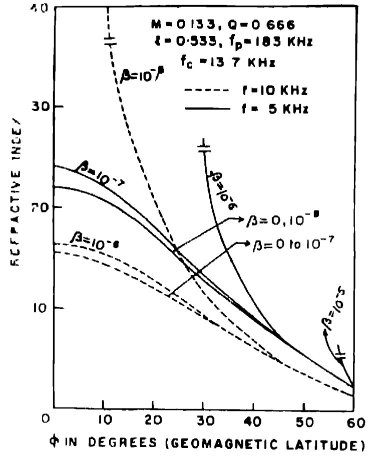

Fig 1 Variation of phase refractive index (μ) with geomagnetic latitude (ϕ) at L -- 1 when the irregularities (β) are not field aligned f 5 kHz, 10 kHz

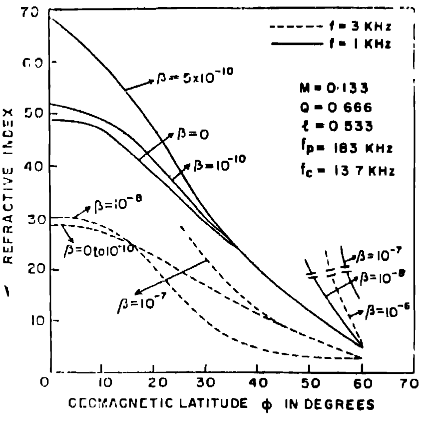

Fig 2 Variation of phase refractive index (μ) with geomagnetic latitude (ϕ) at L — 4 when the irregularities (β) are not field aligned. f = 1 kHz, 3 kHz.

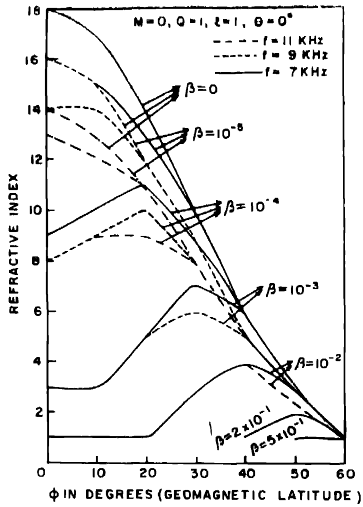

Fig. 3 Variation of phase refractive index (μ) of the wave with geomagnetic latitude (ϕ) at $L = 3$ ($f_p = 183$ kHz, $f_c = 13.7$ kHz) when irregularities (β) are field aligned $f = 7$ kHz, 9 kHz, 11 kHz

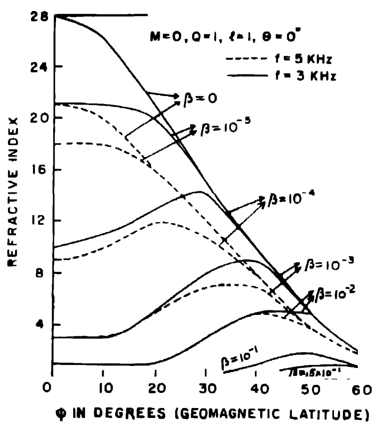

Fig. 4 Variation of phase refractive index (μ) of the wave with geomagnetic latitude (ϕ) at $L = 4$ ($f_p = 184$ kHz, $f_c = 13.7$ kHz) when irregularities (β) are field aligned, $f = 3$ kHz, 5 kHz,

a frequency of 3 kHz the group refractive index shows a flat maximum in the range of 10–20° latitudes. Therefore, the probability of energy propagation of VLF waves becomes less, since group velocity decreases. That is, the guiding of VLF waves becomes poor. The bounce period for 10 kHz wave has been shown in the table 1 with the help of eq. (15).

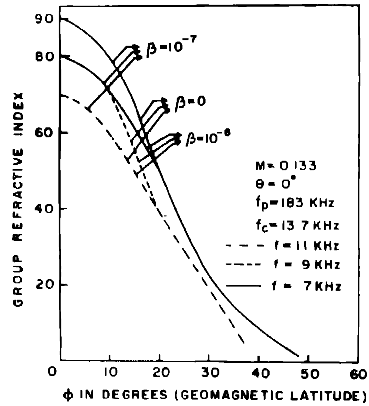

Fig 5 Variation of group refractive index (μ') with geomagnetic latitude (ϕ) at $L = 4$, when irregularities (β) are not field aligned. $f = 7$ kHz, 11 kHz

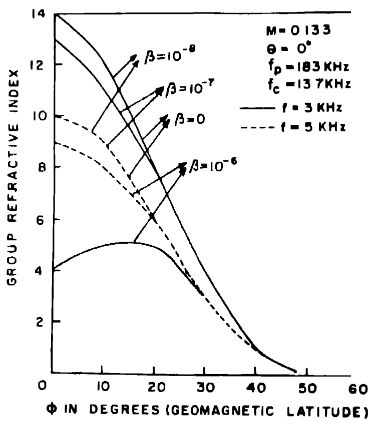

Fig 6. Variation of group refractive index (μ') with geomagnetic latitude (ϕ) at $L = 4$, when irregularities (β) are not field aligned. $f = 3$ kHz, 5 kHz.

Table 1. Bounce period for 10 kHz wave frequency propagating in ducted whistler mode

β	T in seconds			
	$L = 1.3$	$L = 2.0$	$L = 3.0$	$L = 4.0$
0	0.642	1.310	2.169	3.600
10^{-6}	*	*	*	2.596
10^{-7}	*	0.822	1.900	3.480
10^{-8}	0.555	1.291	2.117	3.600

The asterisks (*) in the table shows that for the particular value of β, the VLF wave propagation in the whistler mode is not possible at those L values since the phase refractive index becomes imaginary.

5. Conclusion

The present study concludes that the propagation characteristics of VLF waves in various frequency ranges in presence of field-aligned and non-aligned plasma inhomogeneity are capable of explaining a variety of experimental observations. It is seen that only weak inhomogeneities of $\Delta N/N$ (of the order of 10^{-3}) are needed to guide the whistler waves. The non field aligned inhomogeneities absorb most of the VLF waves. With proper modelling of the magnetosphere (the deformations of geomagnetic field from dipole, plasma density and temperature distributions) it is possible to depict the detailed propagational features of VLF waves. The decisive role of magnetospheric inhomogeneities can be isolated by the detailed study of the role of irregularity present in the ionosphere as well as in the magnetosphere. The ionospheric and magnetospheric irregularities scatter and diffract the incident electromagnetic signals. The diffracted signals when received on the ground produce a diffused trace on whistler spectrograms. The inclusion of the effect of inhomogeneities in the ray tracing theory of magnetospheric reflected whistlers would reveal further interesting features (Walter & Angerami 1969).

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